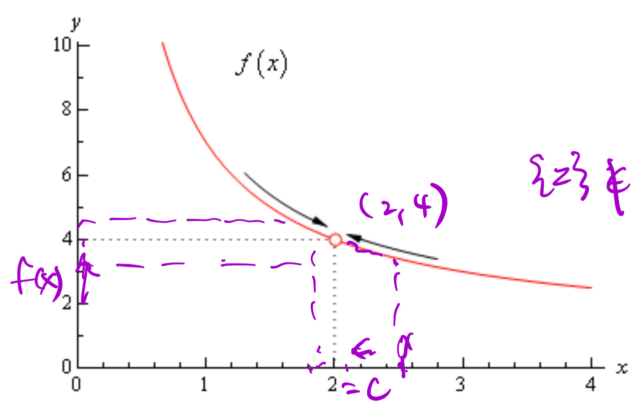
Chapter 2: Limits

Learning Objectives:
(1) Examine the limit concept and general properties of limits.
(2) Compute limits using a variety of techniques.
(3) Compute and use one-sided limits.
(4) Investigate limits involving infinity and "e".

2.1 Limit of a function at one point

(Heuristic) "Definition" 2.1.1. If $f(x)$ gets "closer and closer" to a number L as x gets "closer and closer" to c from *both sides*, then L is called the limit of $f(x)$ as x approaches c, denoted by

$$\lim_{x \to c} f(x) = L.$$



$\{2\} \notin$ domain

$f(2)$ is undefined

but $f(x)$ defined

when $x \neq 2$

but very close

Remark. Limits are defined rigorously via "$\varepsilon - \delta$" language.

Example 2.1.1. Let $f(x) := x + 1$. Find $\lim_{x \to 1} f(x)$

x	0.9	0.99	0.999	1	1.001	1.01	1.1
$f(x)$	1.9	1.99	1.999	2	2.001	2.01	2.1

When x approaches 1 from both sides, $f(x)$ approaches 2. Therefore $\lim_{x \to 1} f(x) = 2$.

$f(c) = \underset{x \to c}{f(x)}$ only when f is "good"

Remark. 1. The table only gives you an intuitive idea, this is not a rigorous proof.
2. Don't think that the limit is always obtained by substituting $x = 1$ into $f(x)$. The limit only depends on the behavior of $f(x)$ near $x = 1$, but not at $x = 1$.

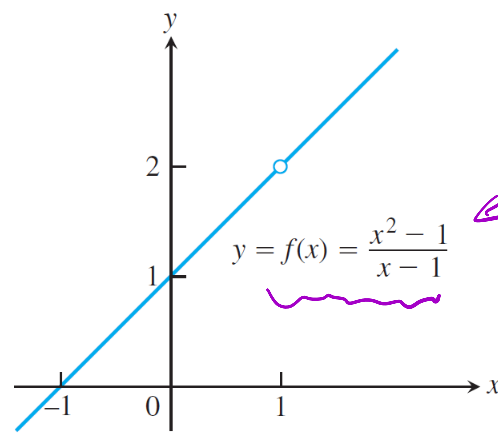
Example 2.1.2. $f(x) = \begin{cases} x+1 & \text{if } x \neq 1, \\ \text{undefined} & \text{if } x = 1. \end{cases}$

$x \to 1$

x	0.9	0.99	0.999	**1**	1.001	1.01	1.1
$f(x)$	1.9	1.99	1.999	**undefined**	2.001	2.01	2.1

When x approaches 1 from both sides, $f(x)$ approaches 2. Therefore $\lim_{x\to 1} f(x) = 2$.

Disregard the value of f at 1, the limit of $f(x)$ when x tends to 1 is always 2.



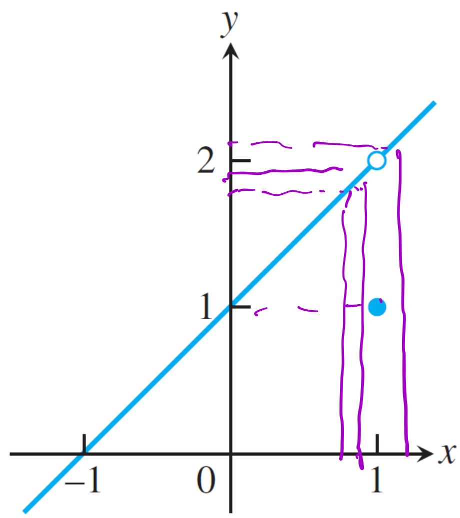
undefined when $x=1$
natural domain
$\mathbb{R}\setminus\{0\}$

$f = \dfrac{x^2-1}{x-1} = \dfrac{(x+1)(x-1)}{x \neq 1}$
when $x \neq 1$ $= x+1$

Example 2.1.3. $f(x) = \begin{cases} x+1 & \text{if } x \neq 1, \\ 1 & \text{if } x = 1. \end{cases}$

x	0.9	0.99	0.999	**1**	1.001	1.01	1.1
$f(x)$	1.9	1.99	1.999	**1**	2.001	2.01	2.1

When x approaches 1 from both sides, $f(x)$ approaches 2. Therefore $\lim_{x\to 1} f(x) = 2$.



$f(1) = 1$

$\neq$

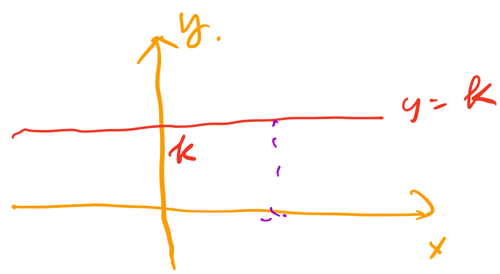
$\lim_{x\to 1} f(1) = 2$

Proposition 1.

1. If $f(x) = k$ is a constant function, then

$$\lim_{x\to c} f(x) = \lim_{x\to c} k = k.$$

For instance, $\lim_{x\to 1} 9 = 9.$ $= \lim_{x\to 0} 9 = 9$

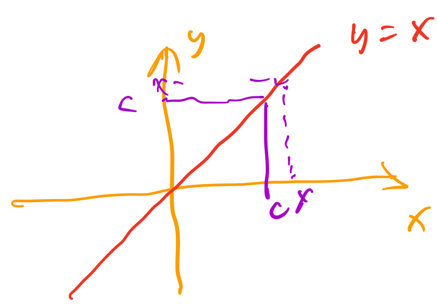


2. If $f(x) = x$, then

$$\lim_{x\to c} f(x) = \lim_{x\to c} x = c.$$

For instance, $\lim_{x\to 3} x = 3.$

$\lim_{x\to -1} x = -1$



Proposition 2. **(Algebraic properties of limits,** $+, -, \times, \div$**)**

If $\lim_{x\to c} f(x)$ and $\lim_{x\to c} g(x)$ both exist (important!), then

1. $\lim_{x\to c}(f(x) + g(x)) = \lim_{x\to c} f(x) + \lim_{x\to c} g(x)$
2. $\lim_{x\to c}(f(x) - g(x)) = \lim_{x\to c} f(x) - \lim_{x\to c} g(x)$
3. $\lim_{x\to c}(f(x)g(x)) = \lim_{x\to c} f(x) \cdot \lim_{x\to c} g(x)$

   Especially, $\lim_{x\to c} kf(x) = k \lim_{x\to c} f(x)$ for any constant k

$\lim_{x\to c}(k f(x)) = \left(\lim_{x\to c} k\right) \cdot \left(\lim_{x\to c} f(x)\right)$

$= k \lim_{x\to c} f(x)$

4. $\lim_{x\to c}\left(\dfrac{f(x)}{g(x)}\right) = \dfrac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)}$ if $\lim_{x\to c} g(x) \neq 0.$

5. $\lim_{x\to c}(f(x))^p = \left[\lim_{x\to c} f(x)\right]^p$ if $\left[\lim_{x\to c} f(x)\right]^p$ exists

Example 2.1.4. Compute the following limits:

$\lim_{x\to1}(2x) = 2\lim_{x\to1} x = 2\cdot 1.$

$\lim_{x\to1} x^3 = \left(\lim_{x\to1} x\right)^3 = 1^3$

1. $\lim_{x\to1}(x^3+2x-5)$
2. $\lim_{x\to2}\dfrac{x^4+x^2-1}{x^2+5}$
3. $\lim_{x\to-2}\sqrt{4x^2-3}$

Solution.

1. $\lim_{x\to1}(x^3+2x-5) = \lim_{x\to1}x^3 + \lim_{x\to1}2x - \lim_{x\to1}5 = 1^3+2\cdot1-5 = -2.$

   $(\lim_{x\to2} x)^4 \cdot (\lim_{x\to2} x)^2$

2. $\lim_{x\to2}\dfrac{x^4+x^2-1}{x^2+5} = \dfrac{\lim_{x\to2}(x^4+x^2-1)}{\lim_{x\to2}(x^2+5)} = \dfrac{\lim_{x\to2}x^4+\lim_{x\to2}x^2-\lim_{x\to2}1}{\lim_{x\to2}x^2+\lim_{x\to2}5} = \dfrac{19}{9}.$

   $2^4+2^2-1=19$

   $(\lim_{x\to2} x)^2+5 = 2^2+5 = 9$

3. $\lim_{x\to-2}\sqrt{4x^2-3} = \sqrt{\lim_{x\to-2}(4x^2-3)} = \sqrt{\lim_{x\to-2}4x^2-\lim_{x\to-2}3} = \sqrt{16-3} = \sqrt{13}.$

   $\lim_{x\to-2}(4x^2-3)^{\frac12} = \left(\lim_{x\to-2}(4x^2-3)\right)^{\frac12} = \left(4\cdot(-2)^2-3\right)^{\frac12} = \sqrt{13}$

∎

Remark. Generalizing the arguments for the first example above: the limit of any polynomial function $P(x)$,

$$\lim_{x\to c}P(x) = P(c).$$

$\frac{1}{x-1}$ undefined when $x=1$

Exercise 2.1.1. Compute the following limits:

$$\lim_{x\to1}\frac{1}{x-1};\qquad \lim_{x\to1}\left(x^2-\frac{3x}{x+5}\right)$$

Example 2.1.5. (Cancelling a common factor)
Find the limit:

$$\lim_{x\to1}\frac{x^2-1}{x^2-3x+2}.$$

undefined when $x=1$.
defined when x is close to 1,
but $\neq 1$

Solution. We can't directly use property of division of limit because the denominator $\lim_{x\to1}(x^2-3x+2) = 1^2-3\times1+2 = 0.$

$$\begin{aligned}\lim_{x\to1}\frac{x^2-1}{x^2-3x+2} &= \lim_{x\to1}\frac{(x-1)(x+1)}{(x-1)(x-2)}\\ &= \lim_{x\to1}\frac{\cancel{(x-1)}(x+1)}{\cancel{(x-1)}(x-2)}\\ &= \lim_{x\to1}\frac{x+1}{x-2}\\ &= \frac{1+1}{1-2} = -2.\end{aligned}$$

$= \dfrac{\lim_{x\to1}(x+1)}{\lim_{x\to1}(x-2)} = \dfrac{2}{-1} = -2$

$\blacksquare$

$x^2+2x-3=0$ when $x=1$

Example 2.1.6. Compute

$$\lim_{x\to 1}\frac{x^3-5x+4}{x^2+2x-3}.$$

← not defined when $x=1$

Solution. Write $p(x)=x^3-5x+4$ and $q(x)=x^2+2x-3$. Because $p(1)=q(1)=0$, $x-1$ is a factor of $p(x)$ and $q(x)$. We obtain

$$p(x)=(x-1)(x^2+x-4) \text{ and } q(x)=(x-1)(x+3).$$

Then

$$\begin{aligned}\lim_{x\to 1}\frac{x^3-5x+4}{x^2+2x-3} &= \lim_{x\to 1}\frac{(x-1)(x^2+x-4)}{(x-1)(x+3)}\\ &= \lim_{x\to 1}\frac{x^2+x-4}{x+3}\\ &= \frac{1^2+1-4}{1+3} = -\frac{1}{2}.\end{aligned}$$

$\blacksquare$

← undefined when $x=1$

Example 2.1.7. (Rationalization)

Let $f:[0,\infty)\backslash\{1\}\to\mathbf{R}$ defined by $f(x)=\frac{\sqrt{x}-1}{x-1}$. Find $\lim_{x\to 1} f(x)$.

$(a-b)(a+b)=a^2-b^2$

$(\sqrt{x}-1)(\sqrt{x}+1)=x-1$

Solution. For $x\neq 1$.

$$\frac{\sqrt{x}-1}{x-1}=\frac{\sqrt{x}-1}{x-1}\cdot\frac{\sqrt{x}+1}{\sqrt{x}+1}=\frac{\cancel{x-1}}{\cancel{(x-1)}(\sqrt{x}+1)}=\frac{1}{\sqrt{x}+1}.$$

Hence

$$\lim_{x\to 1}\frac{\sqrt{x}-1}{x-1}=\lim_{x\to 1}\frac{1}{\sqrt{x}+1}=\frac{1}{2}.$$

$$= \frac{\lim_{x\to 1} 1}{\lim_{x\to 1}(\sqrt{x}+1)} = \frac{1}{\lim_{x\to 1}\sqrt{x}+1} = \frac{1}{\sqrt{1}+1} = \frac{1}{2}$$

$\blacksquare$

Example 2.1.8. (Rationalization and Cancellation)

Find

$$\lim_{x\to 1}\frac{\sqrt{x}-1}{x^2-1}.$$

Solution.

$$\begin{aligned}\lim_{x\to 1}\frac{\sqrt{x}-1}{x^2-1} &= \lim_{x\to 1}\frac{(\sqrt{x}-1)(\sqrt{x}+1)}{(x+1)(x-1)(\sqrt{x}+1)}\\ &= \lim_{x\to 1}\frac{\cancel{x-1}}{(x+1)\cancel{(x-1)}(\sqrt{x}+1)}\\ &= \lim_{x\to 1}\frac{1}{(x+1)(\sqrt{x}+1)} = \frac{1}{4}.\end{aligned}$$

$\blacksquare$

$$= \frac{\lim_{x\to 1} 1}{\lim_{x\to 1}\left((x+1)(\sqrt{x}+1)\right)} = \frac{1}{\left(\lim_{x\to 1}(x+1)\right)\left(\lim_{x\to 1}(\sqrt{x}+1)\right)}$$

$= \frac{1}{2\cdot 2}$

$= \frac{1}{4}$

$a = \sqrt[3]{x} \quad b = 1$

Challenge Question: Let $f : \mathbf{R}\setminus\{1\} \to \mathbf{R}$ defined by $f(x) = \dfrac{\sqrt[3]{x}-1}{x-1}$. $\cdot \dfrac{(x^{\frac{2}{3}}+x^{\frac{1}{3}}+1)}{x^{\frac{2}{3}}+x^{\frac{1}{3}}+1} = \dfrac{\cancel{(x-1)}}{\cancel{(x-1)}(x^{\frac{2}{3}}+x^{\frac{1}{3}}+1)}$

Find $\lim\limits_{x\to 1} f(x)$. $= \lim\limits_{x\to 1} \dfrac{1}{x^{2/3}+x^{\frac{1}{3}}+1} = \cdots$

Hint: $a^3 - b^3 = (a-b)(a^2+ab+b^2)$.

$u = g(x)$

Proposition 3 (Composite functions/change of variables)**.** If $\lim_{x\to c} g(x) = k$ exists and $\lim_{u\to k} f(u)$ exists, then $\lim_{x\to c} f\circ g(x) = \lim_{u\to k} f(u)$.

Example 2.1.9. Redo the last three examples using change of variables.

E.g.

$$\lim_{x\to 1} \frac{\sqrt{x}-1}{x-1}$$

let $u = \sqrt{x}$ when $x > 0$

$\rightsquigarrow u^2 = x$

$$\lim_{x\to 1} u = \lim_{x\to 1} \sqrt{x} = \sqrt{1} = 1$$

$$= \lim_{u\to 1} \frac{u-1}{u^2-1}$$

$$= \lim_{u\to 1} \frac{\cancel{(u-1)}}{\cancel{(u-1)}(u+1)}$$

$$= \lim_{u\to 1} \frac{1}{u+1}$$

$$= \frac{\lim_{u\to 1} 1}{\lim_{u\to 1}(u+1)} = \frac{1}{2}$$

a

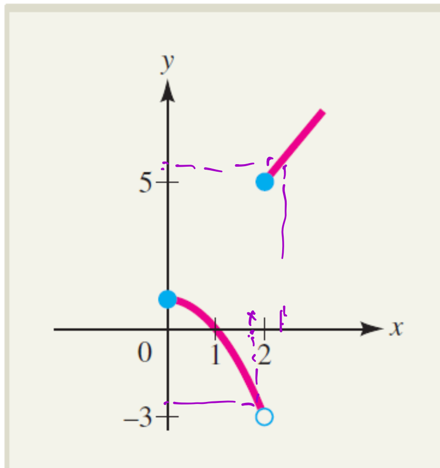
E.g.

$$\lim_{x\to 1} \frac{\sqrt[3]{x}-1}{x-1}$$

: Hint : Let $u = \sqrt[3]{x}$

2.2 One-sided Limits

The following shows the graph of a piecewise function $f(x)$:



As x approaches 2 from the right, $f(x)$ approaches 5 and we write

$$\lim_{x \to 2^+} f(x) = 5. \neq \lim_{x\to 2^-} f(x)$$

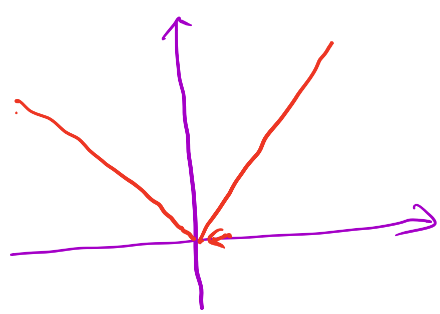
On the other hand, as x approaches 2 from the left, $f(x)$ approaches -3 and we write

$$\lim_{x \to 2^-} f(x) = -3.$$

Limits of these forms are called one-sided limits. The limit is a right-hand limit if the approach is from the right. From the left, it is a left-hand limit.

Definition 2.2.1. If $f(x)$ approaches L as x tends towards c from the left ($x < c$), we write $\lim_{x \to c^-} f(x) = L$. It is called the **left-hand limit** of $f(x)$ at c.
If $f(x)$ approaches L as x tends towards c from the right ($x > c$), we write $\lim_{x \to c^+} f(x) = L$. It is called the **right-hand limit** of $f(x)$ at c.

Example 2.2.1. Recall



$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

$$\lim_{x \to 0^+} |x| = \lim_{x \to 0^+} x = 0.$$

$$\lim_{x \to 0^-} |x| = \lim_{x \to 0^-} (-x) = 0.$$

$\lim_{x\to 0} x = 0 = \lim_{x\to 0^+} x$

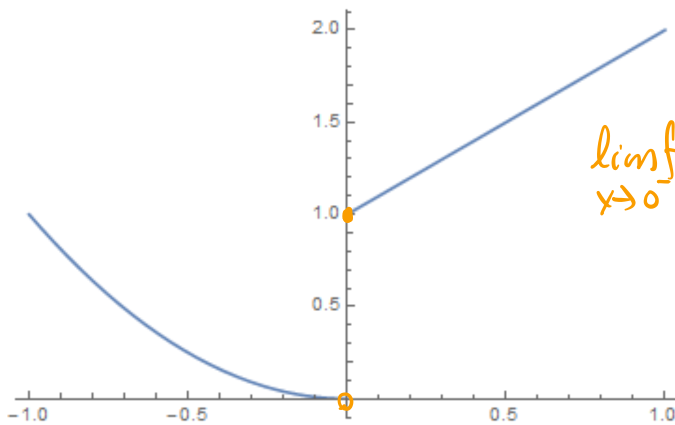
$\lim_{x\to 0^-} (-x) = \lim_{x\to 0} (-x) = 0$

For this case $\lim_{x\to 0^+} |x| = \lim_{x\to 0^-} |x|$. Then $\lim_{x\to 0} |x| = 0$.

Example 2.2.2. Define $f : \mathbf{R} \to \mathbf{R}$,

$$f(x) = \begin{cases} x+1 & \text{if } x \geq 0, \\ x^2 & \text{if } x < 0. \end{cases}$$

$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} (x+1) = \lim_{x\to 0} (x+1) = 1$



$\lim_{x\to 0} f(x) = \lim_{x\to 0^-} x^2 = \lim_{x\to 0} x^2 = 0$

x	-0.1	-0.01	-0.001	0	0.001	0.01	0.1
$f(x)$	10^{-2}	10^{-4}	10^{-6}	1	1.001	1.01	1.1

We have

$$\lim_{x\to 0^+} f(x) = 1.$$

and

$$\lim_{x\to 0^-} f(x) = 0.$$

Remark.

1. The left hand limit or the right hand limit may not be the same.

2. Does $\lim_{x\to 0} f(x)$ exist? No!

Proposition 4.

$\lim_{x\to c} f(x) = L$ if and only if $\lim_{x\to c^-} f(x) = L$ and $\lim_{x\to c^+} f(x) = L$.

(i.e., both left hand limit and right hand limit exist and is equal to L)

Example 2.2.3. Suppose the function

$$f(x) = \begin{cases} x^2+1, & x \geq 1, \\ a, & x < 1. \end{cases}$$

has a limit as x approaches 1. Find the value of a and $\lim_{x\to 1} f(x)$.

Solution. Since $\lim_{x\to 1} f(x)$ exists, we have

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^-} f(x) = \lim_{x\to 1} f(x).$$

$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^-} f(x)$

if $\lim_{x\to 1} f(x)$ exists

And

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+}(x^2+1) = 2, \quad \lim_{x\to 1^-} f(x) = \lim_{x\to 1^-}(a) = a.$$

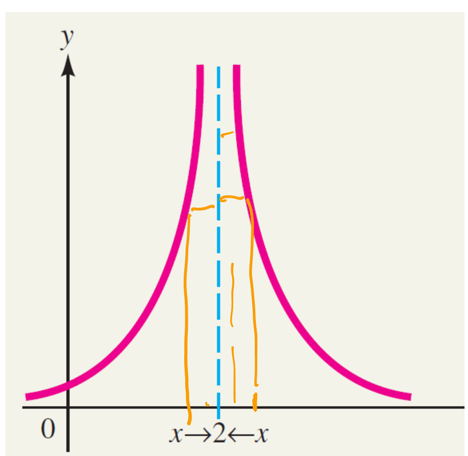
So, $a = 2$, and $\lim_{x\to 1} f(x) = 2$. ■

2.3 Infinite "Limits"

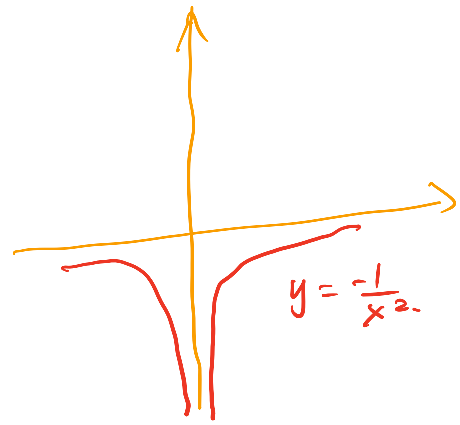
Consider the following limit

$$\lim_{x\to 2} \frac{1}{(x-2)^2}.$$

As x approaches 2, the denominator of the function $f(x) = \dfrac{1}{(x-2)^2}$ approaches 0 and hence the value of $f(x)$ becomes very large.



The function $f(x)$ increases without bound as $x \to 2$ both from left and from right. In this case, the limit *DNE (does not exist)* at $x = 2$, but we express the asymptotic behaviour



of f near 2 *symbolically* as

$$\lim_{x \to 2} \frac{1}{(x-2)^2} = +\infty.$$

Remark. $+\infty$ is just a symbol, not a real number.

Example 2.3.1.

$$\lim_{x \to 0} \frac{-1}{x^2} = -\infty.$$

Definition 2.3.1. We say that $\lim_{x \to c} f(x)$ is an infinite limit if $f(x)$ increases or decreases without bound as $x \to c$.
If $f(x)$ increases without bound as $x \to c$, we write

$$\lim_{x \to c} f(x) = +\infty.$$

If $f(x)$ decreases without bound as $x \to c$, then

$$\lim_{x \to c} f(x) = -\infty.$$

Example 2.3.2. Evaluate

$$\lim_{x \to 2^+} \frac{x-3}{x^2-4} \text{ and } \lim_{x \to 2^-} \frac{x-3}{x^2-4}.$$

Solution.

$$\lim_{x \to 2^+} \frac{x-3}{x^2-4} = \lim_{x \to 2^+} \frac{x-3}{(x-2)(x+2)} = -\infty$$

since as $x \to 2^+$, we have $x^2 - 4 = (x-2)(x+2) \to 0^+$ and $x - 3 \to -1^+$.

$$\lim_{x \to 2^-} \frac{x-3}{x^2-4} = \lim_{x \to 2^-} \frac{x-3}{(x-2)(x+2)} = +\infty$$

since as $x \to 2^-$, we have $x^2 - 4 = (x-2)(x+2) \to 0^-$ and $x - 3 \to -1^-$.

■

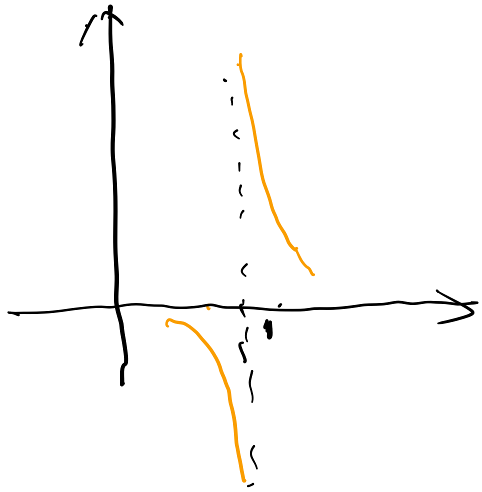
Exercise 2.3.1. Find

$$\lim_{x \to \pi/2} \tan x; \qquad \lim_{x \to \pi/2^-} \tan x; \qquad \lim_{x \to \pi/2^+} \tan x; \qquad \lim_{x \to 0^+} \ln x.$$

E.g.,

$$\lim_{x \to 1} \frac{1}{x-1}$$

limit does not exist.



E.g.,

$$\lim_{x \to 1} \frac{x^3-5x+4}{x^2+2x-3}$$

Fact: $P(x)$: polynomial
$P(c)=0$
then $(x-c) \mid P(x)$
$P(x) = (x-c)\,Q(x)$
another polynomial found by long division

when $x=1$

$$\begin{cases} x^3-5x+4 = 0 \\ x^2+2x-3 = 0 \end{cases}$$

so $x^3-5x+4 = (x-1)(x^2+x-4)$

similarly $x^2+2x-3 = (x-1)(x+3)$

$$\begin{array}{r} x^2 + x - 4 \\ x-1 \overline{\smash{)}\, x^3 \qquad -5x+4} \\ \underline{x^3 - x^2 \qquad\qquad} \\ x^2 - 5x + 4 \\ \underline{x^2 - x \qquad} \\ -4x+4 \\ \underline{-4x+4} \\ 0 \end{array}$$

$$\lim_{x \to 1} \frac{\cancel{(x-1)}(x^2+x-4)}{\cancel{(x-1)}(x+3)}$$

$$= \lim_{x \to 1} \frac{x^2+x-4}{x+3}$$

$$= \frac{\lim_{x \to 1}(x^2+x-4)}{\lim_{x \to 1}(x+3)} = \frac{1^2+1-4}{1+3} = \frac{-2}{4} = -\frac{1}{2} \quad \square$$